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# Verifiable Necessary and Sufficient Conditions for Openness and Regularity of Set-Valued and Single-Valued Maps

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We provide several equivalences to the regularity of closed set-valued maps around a point in general metric settings. In particular, an easy to verify approximate openness notion is shown equivalent to regularity. A simple specialization of our theorem strengthens Frankowska's novel "open mapping principle." 

## 1. INTRODUCTION

Regularity of both single-valued and set-valued maps has been studied by many authors. It links openness and inversion properties so closely that investigation of regular behavior becomes a useful approach in studying perturbed optimization problems, stability, and controllability (see [7, 10, 14, 18–20, 23, 24] and elsewhere).

In this paper we study regularity properties of closed set-valued maps in metric spaces. We prove that our regularity property is actually equivalent to openness, to approximate openness, to an inversion property, and to a high order variation property (in an appropriate sense). Our theorems provide a unified way to establish many results.

Recently, Khanh [16] extending work of Ptàk [21] and Dolecki [11] proved a so-called induction thereom. Using it as a basic tool he proved some general open mapping theorem. We could largely derive our results from his, but we prefer a direct proof, based on Ekeland's powerful $\varepsilon$-variational principle [2], which is more in the spirit of non-smooth analysis and which gives our results in a form accessible for application. (See Borwein [7], where regularity is systematically applied to establish inversion, tangency and stability results.)

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The organization of the paper is as follows. In Section 2 we introduce general definitions of $\delta$-regularity, $\delta$-openness, and $\delta$-approximate openness and establish their equivalence. In Section 3 we show that regularity is indeed an inversion property. Several celebrated results are rederived as easy consequences of our main theorems. In our final section we study the connection between regularity and differentiation of set-valued maps. Frankowska's open mapping principle [12] is strengthened and one more property is added to the equivalence of regularity provided that the range space of the map is finite dimensional.

Throughout the paper we shall frequently use the notations

$$d(x, A) := \inf\{d(x, a) \mid a \in A\};$$

$$B_\varepsilon(x_0) := \{x \in X \mid d(x, x_0) \leqslant \varepsilon\};$$

$$B_\varepsilon(A) := \{x \in X \mid d(x, A) \leqslant \varepsilon\};$$

and if $X$ is a normed space,

$$B_X := \{x \in X \mid \|x\| \leqslant 1\}.$$

Also

$$\underline{\lim} := \lim\inf;$$

$$\overline{\lim} := \lim\sup$$

## 2. The Main Theorem

Let $\Omega: X \to Y$ be a set-valued map and $y_0 \in \Omega(x_0)$ be given, with $X$ and $Y$ being metric spaces. Let $\delta: \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly monotone continuous function with $\delta(0) = 0$. We introduce the following definitions.

2.1. Definitions. (i) $\Omega$ is *$\delta$-open around* $(x_0, y_0)$ if there exists a neighbourhood $U$ of $x_0$ and a neighbourhood $W$ of $y_0$ such that

$$B_{\delta(t)}(z) \subset \Omega(B_t(x)), \tag{2.1}$$

for all $x$ in $U$, all $z$ in $W \cap \Omega(x)$ and all $t > 0$ with $B_t(x_0) \subset U$.

(ii) $\Omega$ is *approximately $\delta$-open around* $(x_0, y_0)$ if there exist some non-negative function $\alpha: \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t \downarrow 0}(\delta^{-1}(\alpha(t))/t) < 1$, a neighbourhood $U$ of $x_0$, and a neighbourhood $W$ of $y_0$, such that for all $x$ in $U$, all $z$ in $W \cap \Omega(x)$, and all $t > 0$ with $B_t(x_0) \subset U$,

$$B_{\delta(t)}(z) \subset \overline{B_{\alpha(t)}[\Omega(B_t(x))]}. \tag{2.2}$$

(iii) $\Omega$ is *$\delta$-regular around* $(x_0, y_0)$ if there exist a constant $K>0$, a neighbourhood $U$ of $x_0$, a neighbourhood $W$ of $y_0$, such that for all $z$ in $W$ and all $x$ in $U$ with $\Omega(x) \cap W \neq \varnothing$

$$d[x, \Omega^{-1}(z)] \leqslant K\delta^{-1}[d(z, \Omega(x))]. \tag{2.3}$$

2.2. THEOREM. *Let $X$ and $Y$ be metric spaces and $\Omega: X \to Y$ a set-valued map with complete graph. Then for given $y_0 \in \Omega(x_0)$ the following are true:*

(i) *If $\Omega$ is $\delta$-open around $(x_0, y_0)$, then $\Omega$ is approximately $\delta$-open around $(x_0, y_0)$.*

(ii) *If $\Omega$ is approximately $\delta'$-open around $(x_0, y_0)$, then $\Omega$ is $\delta$-regular around $(x_0, y_0)$ for any $\delta \leqslant \delta'$ with $\overline{\lim}_{t \downarrow 0} (\delta(t)/t) < \infty$.*

(iii) *If $\Omega$ is $\delta$-regular around $(x_0, y_0)$, then $\Omega$ is $\tilde{\delta}$-open around $(x_0, y_0)$, where $\tilde{\delta} = \delta(c_0 t)$ for some $c_0 > 0$ and hence for all $0 < c < c_0$.*

*Proof.* (i) is clear because for $\alpha(t) \geqslant 0$

$$B_{\alpha(t)}(z) \subset \Omega(B_t(x)) \subset \overline{B_{\alpha(t)}[\Omega(B_t(x))]}.$$

Now we prove (ii). Consider the failure of (2.3). Then there is some strictly monotone function $\delta$ with $\delta(0)=0$ and $\delta \leqslant \delta'$ such that for this $\delta$, we can find sequences $(x'_n) \subset X$, $(y'_n) \subset Y$ with $x'_n \to x_0$, $y'_n \to y_0$, and $d(y_0, \Omega(x'_n)) \to 0$ while for $n = 1, 2, \ldots$,

$$d[x'_n, \Omega^{-1}(y'_n)] > n\delta^{-1}[d(y'_n, \Omega(x'_n))]. \tag{2.4}$$

Define $f_n: \operatorname{Gr} \Omega \subset X \times Y \to \mathbb{R}$ by

$$f_n(x, z) = \delta^{-1}[d(z, y'_n)], \qquad \text{if} \quad (x, z) \in \operatorname{Gr} \Omega.$$

Then $f_n$ is continuous on $\operatorname{Gr} \Omega$. Let

$$\varepsilon_n := f_n(x'_n, z'_n) = \delta^{-1}[d(y'_n, z'_n)],$$

where $z'_n \in \Omega(x'_n)$ is chosen so that $d(y'_n, \Omega(x'_n)) + 1/n > d(y'_n, z'_n)$ and $d[x'_n, \Omega^{-1}(y'_n)] > n\delta^{-1}[d(y'_n, z'_n)]$.

It follows from (2.4) that $\varepsilon_n > 0$ because $d(y'_n, z'_n) \geqslant d(y'_n, \Omega(x'_n))$. Note also that $\varepsilon_n$ approaches zero as $n$ goes to infinity because

$$d(y'_n, z'_n) \leqslant d(y'_n, y_0) + d(y_0, z'_n) \leqslant d(y'_n, y_0) + d(y_0, \Omega(x'_n)) + 1/n.$$

It is evident that

$$f_n(x'_n, z'_n) \leqslant \inf_{\operatorname{Gr} \Omega} f + \varepsilon_n.$$

Let $\lambda_n := \min(n\varepsilon_n, \sqrt{\varepsilon_n})$. Then $\lambda_n > 0$ and $\lambda_n \to 0$. Since $\operatorname{Gr}\Omega$ is complete and $f_n$ is continuous, Ekeland's $\varepsilon$-variational principle applied to $f_n$ on the metric space $\operatorname{Gr}\Omega$ with

$$d[(x_1, y_1), (x_1, y_2)] := d(x_1, x_2) + d(y_1, y_2)$$

asserts the existence of $(x_n, z_n)$ in $\operatorname{Gr}\Omega$ such that

(a) $d[(x_n, z_n), (x'_n, z'_n)] \leqslant \lambda_n$;

(b) $f_n(x_n, z_n)) \leqslant f_n(x'_n, z'_n) = \varepsilon_n$;

and

(c) $f_n(x, z) - f_n(x_n, z_n) \geqslant -\rho_n d[(x_n, z_n), (x, z)]$, if $z \in \Omega(x)$,

where $\rho_n := \varepsilon_n/\lambda_n = \max(\sqrt{\varepsilon_n}, 1/n)$.

From (a) we see that $(x_n, y_n) \to (x_0, y_0)$ because

$$d[(x_n, z_n), (x_0, y_0)] \leqslant d[(x'_n, z'_n), (x_0, y_0)] + \lambda_n.$$

Clearly, $\Omega(x_n)$ is nonempty and (c) shows that for $z \in \Omega(x)$

$$\delta^{-1}[d(z, y'_n)] - \delta^{-1}[d(z_n, y'_n)] \geqslant -\rho_n[d(x_n, x) + d(z_n, z)]. \tag{2.5}$$

Denote $\delta^{-1}[d(z_n, y'_n)]$ by $t_n$. Then $t_n > 0$ since

$$d(x_n, x'_n) \leqslant \lambda_n \leqslant n\varepsilon_n < d(x'_n, \Omega^{-1}(y'_n))$$

so that $y'_n \notin \Omega(x_n)$. Also $t_n \to 0$. Take $\hat{x}$ in $B_{t_n}(x_n)$ arbitrarily. Replace $(x, z)$ by $(\hat{x}, z) \in \operatorname{Gr}\Omega$ in (2.5). Then $d(x_n, \hat{x}) \leqslant t_n$ and

$$\begin{aligned}\delta^{-1}[d(z, y'_n)] - t_n &\geqslant -\rho_n[t_n + d(z_n, z)] \\ &\geqslant -\rho_n[t_n + d(y'_n, z_n) + d(y'_n, z)].\end{aligned}$$

Hence

$$\frac{\delta^{-1}[d(z, y'_n)]}{t_n} - 1 \geqslant -\rho_n\left[1 + \frac{d(y'_n, z_n)}{t_n} + \frac{d(y'_n, z)}{t_n}\right],$$

or

$$\frac{\delta^{-1}[d(z, y'_n)]}{t_n} + \rho_n \frac{d(z, y'_n)}{t_n} \cdot \frac{d(y'_n, z_n)}{d(y'_n, z_n)} \geqslant 1 - \rho_n\left[1 + \frac{d(y'_n, z_n)}{t_n}\right].$$

That is,

$$\frac{\delta^{-1}[d(z, y'_n)]}{t_n} + \rho_n \frac{\delta(t_n)}{t_n} \cdot \frac{d(z, y'_n)}{\delta(t_n)} \geqslant 1 - \rho_n\left[1 + \frac{\delta(t_n)}{t_n}\right].$$

Since we assume that $\overline{\lim}_{t\downarrow 0}(\delta(t)/t)$ is finite and since both $\rho_n$ and $t_n$ go to zero, we have $\rho_n \cdot \delta(t_n)/t_n \to 0$.

Now write $\sigma_n = \rho_n\delta(t_n)/t_n$ and $\mu_n := \rho_n + \sigma_n$. Then both $\sigma_n$ and $\mu_n$ go to zero and

$$\frac{\delta^{-1}[d(z, y'_n)]}{t_n} + \sigma_n \frac{d(z, y'_n)}{\delta(t_n)} \geqslant 1 - \mu_n.$$

Note that $\delta^{-1}$ is a continuous strictly monotone function, and $z$ is chosen from $\Omega(\hat{x}) \subset \Omega(B_{t_n}(x_n))$ arbitrarily; hence we can assert from above that

$$\frac{\inf_{z\in\Omega_n} \delta^{-1}[d(z, y'_n)]}{t_n} + \sigma_n \frac{\inf_{z\in\Omega_n} d(z, y'_n)}{\delta(t_n)} \geqslant 1 - \mu_n,$$

where $\Omega_n := \Omega(B_{t_n}(x_n))$. Therefore

$$\frac{\delta^{-1}[d(y'_n, \Omega_n)]}{t_n} + \sigma_n \frac{d(y'_n, \Omega_n)}{\delta(t_n)} \geqslant 1 - \mu_n.$$

If $d(y'_n, \Omega_n)/\delta(t_n) \to \infty$, then there exists $N > 0$ such that for all $n \geqslant N$, we have

$$d(y'_n, \Omega_n) \geqslant \delta(t_n).$$

In other words, for any $\varepsilon > 0$ and $n \geqslant N$,

$$\frac{\delta^{-1}[(y'_n, \Omega_n)]}{t_n} \geqslant 1 > 1 - \varepsilon.$$

If, on the other hand, $d(y'_n, \Omega_n)/t_n$ is bounded by some $M > 0$ for all $n = 1, 2, \ldots$, then

$$\frac{\delta^{-1}[d(y'_n, \Omega_n)]}{t_n} + \sigma_n M \geqslant 1 - \mu_n.$$

Hence, in both cases for any $\varepsilon > 0$, we can find $N(\varepsilon) > 0$ such that for all $n \geqslant N(\varepsilon)$

$$\frac{\delta^{-1}[d(y'_n, \Omega_n)]}{t_n} \geqslant 1 - \varepsilon. \tag{2.6}$$

Now $\Omega$ is approximately $\delta'$-open around $(x_0, y_0)$, so

$$B_{\delta'(t_n)}(z_n) \subset \overline{B_{\alpha(t_n)}(\Omega_n)}.$$

Note that $d(y'_n, z_n) = \delta(t_n) \leqslant \delta'(t_n)$, hence

$$d(y'_n, \Omega_n) \leqslant \alpha(t_n) + d(B_{\delta'(t_n)}(z_n), y'_n) = \alpha(t_n).$$

It follows from (2.6) that

$$\alpha(t_n) \geqslant d(y'_n, \Omega_n) \geqslant \delta[(1-\varepsilon)\, t_n].$$

This implies that

$$\delta^{-1}[\alpha(t_n)] \geqslant (1-\varepsilon)\, t_n;$$

or

$$\varliminf_{t_n \downarrow 0} \frac{\delta^{-1}[\alpha(t_n)}{t_n} \geqslant 1,$$

which contradicts our assumption that $\varliminf_{t \to 0} \delta^{-1}[\alpha(t)]/t < 1$.

Finally, we verify (iii).

Suppose that $\Omega$ is $\delta$-regular around $(x_0, y_0)$. Then there exist $\varepsilon > 0$ and $K > 0$ such that for all $x$ in $B_\varepsilon(x_0)$ with $\Omega(x) \cap B_\varepsilon(y_0) \neq \varnothing$ and for all $z$ in $B_\varepsilon(y_0)$

$$d[x, \Omega^{-1}(z)] \leqslant K\delta^{-1}[d(x, \Omega(x))].$$

Take $z_1 \in \Omega(x_1)$ with $d(x_1, x_0) \leqslant \varepsilon/2$ and $d(z_1, y_0) \leqslant \varepsilon/2$. Then $d(y_0, \Omega(x_1)) \leqslant d(y_0, z_1) \leqslant \varepsilon/2$. For each $z$ in $B_{\varepsilon/2}(z_1)$, by $\delta$-regularity of $\Omega$,

$$d[x_1, \Omega^{-1}(z)] \leqslant K\delta^{-1}[d(z, \Omega(x_1))].$$

Hence for any positive number $K' > K$ one can find an $x$ in $\Omega^{-1}(z)$ such that

$$d(x, x_1) \leqslant K'\delta^{-1}[d(z, \Omega(x_1))] \leqslant K'\delta^{-1}[d(z, z_1)].$$

Let $t$ be so small that $\delta(t/K') \leqslant \varepsilon/2$. Then

$$B_{\delta(t/K')}(z_1) \subset B_{\varepsilon/2}(z_1),$$

and for each $z$ in $B_{\delta(t/K')}(z_1)$ there exists an $x$ in $\Omega^{-1}(z)$ with

$$d(x, x_1) \leqslant K'\delta^{-1}[d(z, z_1)] \leqslant t.$$

Thus, $x \in \Omega^{-1}(z) \cap B_t(x_1)$ and $z \in \Omega(x) \subset \Omega(B_t(x_1))$. Therefore

$$B_{\delta(t/K')}(z_1) \subset \Omega[B_t(x_1)]$$

for each $x_1 \in B_{\varepsilon/2}(x_0)$ and $z_1 \in R_{\varepsilon/2}(y_0)$ with $z_1 \in \Omega(x_1)$. That is to say, $\Omega$ is $\tilde{\delta}$-open around $(x_0, y_0)$ if we let $\tilde{\delta}(t) := \delta(t/K')$. ∎

As an application of Theorem 2.2 we give the following corollary which essentially extends a result of Bourbaki [4, p. 35, Vol. 1, Lemma 2].

2.3. COROLLARY. *Let $X$ and $Y$ be metric spaces, $\Omega: X \to Y$ a set-valued map with complete graph. Suppose for every $x_0$ in $X$ and any $y_0 \in \Omega(x_0)$ there exist a neighbourhood $U$ of $x_0$, a neighbourhood $W$ of $y_0$, and a strictly positive function $\delta: (0, \infty) \to (0, \infty)$ such that for all $x$ in $U$, all $y$ in $W \cap \Omega(x)$, and all $t > 0$ with $B_t(x_0) \subset U$,*

$$B_{\delta(t)}(y) \subset \overline{\Omega[B_t(x)]}. \tag{2.7}$$

*Then* (i) *$\Omega$ is $\tilde{\delta}$-open around $(x_0, y_0)$ for some strictly monotone continuous function $\tilde{\delta} \leqslant \delta$ with $\tilde{\delta}(0) = 0$.*

(ii) *There exists a neighbourhood $U'$ of $x_0$, a neighbourhood $W'$ of $y_0$, and some $t_0 > 0$ such that for all $\varepsilon > 0$, all $x$ in $U'$, all $y$ in $W' \cap \Omega(x)$, and all $0 < t < t_0$,*

$$B_{\delta(t)}(y) \subset \Omega(B_{t+\varepsilon}(x)). \tag{2.8}$$

*Proof.* (i) Define $\delta_1(t) := \frac{1}{2}\min\{t, \sup_{0 \leqslant s \leqslant t} \delta(s)\}$, $t \geqslant 0$. Let $\{a_n\}$ be a sequence of real numbers strictly decreasing to 0. That is $a_n < a_{n-1}$ for $n = 1, 2, \ldots$ and $\lim_{n \to \infty} a_n = 0$. Let

$$\hat{\delta}(t) := \begin{cases} \delta_1(a_{n+1}) & \text{if } \ t = a_n \\ \delta_1(a_n) + \left[\dfrac{\delta_1(a_n) - \delta_1(a_{n+1})}{a_n - a_{n+1}}\right](t - a_n) & \text{if } \ a_n < t < a_{n-1}. \end{cases}$$

Then for sufficient small $t$, $\hat{\delta}(t)$ is a strictly monotone continuous function with $\hat{\delta} \leqslant \delta_1 \leqslant \delta$ and with $\hat{\delta}(0) = 0$. Now (2.7) implies that

$$B_{\hat{\delta}(t)}(y) \subset \overline{\Omega[B_t(x)]}$$

which shows that $\Omega$ is approximately $\hat{\delta}$-open around $(x_0, y_0)$ (with $\alpha(t) \equiv 0$). Theorem 2.2 asserts that $\Omega$ is $\tilde{\delta}$-open around $(x_0, y_0)$ for some strictly monotone continuous function $\tilde{\delta} \leqslant \hat{\delta}$. Without loss of generality we may assume that $U = B_\rho(x_0)$; $W = B_\rho(x_0)$ and for all $x$ in $U$, all $y$ in $\Omega(x) \cap W$, and all $0 < t < \rho$ we have

$$B_{\tilde{\delta}(t)}(y) \subset \Omega(B_t(x)). \tag{2.9}$$

(ii) Let $U' := B_{\rho/2}(x_0)$ and $W' := B_{\rho/2}(y_0)$. Now (2.7) implies that for all $x$ in $U'$, all $y$ in $\Omega(x) \cap W'$, and all $0 < t < \rho/2$,

$$B_{\delta(t)}(y) \subset B_\alpha[\Omega(B_t(x))]$$

for every $\alpha > 0$. Fix $\varepsilon > 0$ and take $y' \in B_{\delta(t)}(y)$ arbitrarily. We can find for $t \leqslant \varepsilon$ a $v \in B_t(x)$ and a $w \in \Omega(v)$ such that $d(y', w) \leqslant \alpha$. Let $\alpha = \min\{\bar{\delta}(\varepsilon), \rho - \delta(t_0)\}$. Then $w \in W' \subset W$ and $v \in U$. Moreover,

$$y' \in B_\alpha(w) \subset B_{\bar{\delta}(\varepsilon)}(w) \subset \Omega(B_\varepsilon(v)) \subset \Omega(B_{t+\varepsilon}(x)),$$

according to (2.9). Therefore (2.8) holds. ∎

2.4. *Remarks.* (i) The standard open mapping theorem for a closed single-valued linear map $T$ between $F$-spaces (a topological vector space is an $F$-space if its topology is induced by a complete invariant metric) is an immediate consequence of Corollary 2.3. Indeed, the Baire category theorem will provide $B_{\delta(t)}(y) \subset \overline{T(B_t(x))}$ and Corollary 2.3(i) will assert that $T$ is open around $(x, T(x))$.

(ii) When $\Omega$ is a continuous single-valued map and (2.7) is valid for all $x_0$ in $X$ with $U$ being $X$ and $W$ being $Y$ then Corollary 2.3(ii) is Bourbaki's result.

(iii) Ptàk's closed graph theorem [21] is similar to Corollary 2.3 with the assumption only that $X$ is complete. However, the conclusions of Corollary 2.3 are equivalent to our regularity property. Therefore it is more useful and easier to be applied than Ptàk's theorem.

The most interesting candidate for our strictly monotone continuous function $\delta(t)$ is $ct^r$, where $r \geqslant 1$ and $c > 0$. We shall say a set-valued map $\Omega$ is *open(regular) at rate r around* $(x_0, y_0)$ if there exist some $r \geqslant 1$ and some $c > 0$ such that $\Omega$ is $\delta$-open (regular) around $(x_0, y_0)$ for $\delta = ct^r$. We shall also say that $\Omega$ is *approximately open at rate r around* $(x_0, y_0)$ if there exist $r \geqslant 1$, some $c > 0$, and some $0 \leqslant \gamma < c$ such that $\Omega$ is approximately $\delta$-open for $\delta(t) = ct^r$ and $\alpha(t) = \gamma t^r$. Note that $\delta(t) = ct^r$ satisfies the relation $\underline{\lim}_{t \downarrow 0} (\delta(t)/t) < \infty$ and when $\alpha(t) = \gamma t^r$ and $\delta(t) = ct^r$ with $0 \leqslant \gamma < 0$ we have

$$\underline{\lim}_{t \downarrow 0} \frac{\delta^{-1}[\alpha(t)]}{t} < 1.$$

Therefore we can rephrase Theorem 2.2 as follows.

2.5. Theorem. *Let $X$ and $Y$ be metric spaces and $\Omega : X \to Y$ a set-valued map with complete graph. Then for given $y_0 \in \Omega(x_0)$ the following are equivalent*:

(i) *$\Omega$ is open at rate $r$ around* $(x_0, y_0)$;

(ii) *$\Omega$ is approximately open at rate $r$ around* $(x_0, y_0)$;

(iii) *$\Omega$ is regular at rate $r$ around* $(x_0, y_0)$.

## 3. An Inversion Theorem and Some Applications

One of the good features of Theorem 2.2 is that it enables us to verify the approximate openness of a set-valued map when we want to verify the regularity or openneess of the map. This is usually much easier than directly establishing the latter as we shall demonstrate in this section and in the sequel, by rederiving several celebrated results. First, we explore further the intrinsic nature of $\delta$-regularity of a closed set-valued map. It turns out, as one expects, that $\delta$-regularity of a closed set-valued map $\Omega$ is equivalent to a $\delta^{-1}$-*LSC* property of $\Omega^{-1}$—an inversion feature of the map $\Omega$. Here, as always, $x \in \Omega^{-1}(y)$ if and only if $y \in \Omega(x)$.

Analogously to the definition of pseudo-Lipschitzness for set-valued maps introduced in Rubin and Ekeland [2] we now insert the following:

3.1. Definition. Let $X$ and $Y$ be metric spaces and $\Omega: X \to Y$ be a set-valued map. Suppose $\delta: \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly monotone (continuous) function and $y_0 \in \Omega(x_0)$. We say $\Omega^{-1}$ is $\delta^{-1}$-*LSC around* $(y_0, x_0)$ if there exist a neighbourhood $W$ of $y_0$ and a neighbourhood $U$ of $x_0$ such that

(a) $\Omega^{-1}(y) \cap U \neq \varnothing, \forall y \in W;$ (3.1)

(b) there exists $K > 0$ with the property that for every $y_1$ and $y_2$ in $W$,

$$\Omega^{-1}(y_1) \cap U \subset B_{K\delta^{-1}[d(y_1, y_2)]}(\Omega^{-1}(y_2) \cap U) \tag{3.2}$$

when $\delta(t) := ct^r$ for some $c > 0$, $r > 0$, then a map which is $\delta^{-1} - LSC$ around $(y_0, x_0)$ is called *pseudo Hölder at rate $r$ around* $(y_0, x_0)$. When $r = 1$, it is called *pseudo Lipschitz around* $(y_0, x_0)$.

3.2. Theorem. *Let $\Omega$ be a closed set--valued map between metric spaces $X$ and $Y$. Let $\delta$ be as described in Definition* 3.1 *and $y_0 \in \Omega(x_0)$ be given. Then $\Omega$ is $\delta$-regular around $(x_0, y_0)$ if and only if $\Omega^{-1}$ is $\delta^{-1} - LSC$ around* $(y_0, x_0)$.

*Proof.* We assume first that $\Omega$ is $\delta$-regular around $(x_0, , y_0)$. Then there exist $K > 0$, $\varepsilon_0 > 0$, $\varepsilon_1 > 0$ such that for all $y$ in $B_{\varepsilon_1}(y_0)$ and all $x$ in $B_{\varepsilon_0}(x_0)$ with $\Omega(x) \cap B_{\varepsilon_1}(y_0) \neq \varnothing$

$$d[x, \Omega^{-1}(y)] \leqslant K\delta^{-1}[d(y, \Omega(x))]. \tag{3.3}$$

Since $x_0$ is in $B_{\varepsilon_0}(x_0)$ and $y_0$ belongs to $\Omega(x_0) \cap B_{\varepsilon_1}(y_0)$, we clearly have, for all $y$ in $B_{\varepsilon_1}(y_0)$,

$$\begin{aligned} d[x_0, \Omega^{-1}(y)] &\leqslant K\delta^{-1}[d(y, \Omega(x_0))] \\ &\leqslant K\delta^{-1}[d(y, y_0)]. \end{aligned}$$

Hence

$$d(x_0, \Omega^{-1}(y)) \leqslant K\delta^{-1}(\varepsilon_1), \qquad \forall y \in B_{\varepsilon_1}(y_0).$$

Choose $\hat{\varepsilon}_1 < \varepsilon_1$ so small that $K\delta^{-1}(\hat{\varepsilon}_1) \leqslant \varepsilon_0$ and denote $B_{K\delta^{-1}(\hat{\varepsilon}_1)}(x_0)$ by $U$ and $B_{\hat{\varepsilon}_1}(y_0)$ by $W$. Then

$$\Omega^{-1}(y) \cap U \neq \varnothing, \qquad \forall y \in W. \tag{3.4}$$

Pick $y_1, y_2$ arbitrarily from $W$. It follows from (3.4) that $\Omega^{-1}(y_1) \cap U$ and $\Omega^{-1}(y_2) \cap U$ are nonempty. Now for each $x_1$ in $\Omega^{-1}(y_1) \cap U$ we have by (3.3),

$$d[x_1, \Omega^{-1}(y)] \leqslant K\delta^{-1}[d(y, \Omega(x_1))], \qquad \forall y \in W.$$

Thus,

$$d[x_1, \Omega^{-1}(y_2)] \leqslant K\delta^{-1}[d(y_2, \Omega(x_1))].$$

Now for any $K' > K$, we can find an $x_2$ in $\Omega^{-1}(y_2) \cap U$ with the property that

$$d(x_1, x_2) \leqslant K'\delta^{-1}[d(y_2, \Omega(x_1)] \leqslant K'\delta^{-1}[d(y_2, y_1)].$$

Therefore,

$$\Omega^{-1}(y_1) \cap U \subset B_{K'\delta^{-1}[d(y_1, y_2)]}[\Omega^{-1}(y_2) \cap U].$$

Next, we assume that $\Omega^{-1}$ is $\delta^{-1}$-*LSC* around $(y_0, x_0)$. Let $W = B_\varepsilon(y_0)$ and $U = B_\rho(x_0)$ be the neighbourhoods of $y_0$ and $x_0$, respectively, satisfying (3.1) and (3.2). Denote $B_{\varepsilon/3}(y_0)$ by $\tilde{W}$. Then there exists $\tilde{U} \subset U$, a neighbourhood of $x_0$, such that for all $y$ in $\tilde{W}$, $\Omega^{-1}(y) \cap \tilde{U} \neq \varnothing$. Moreover, for all $x$ in $\tilde{U}$ and all $y$ in $\tilde{W}$,

$$d(y, \Omega(x)) = d(y, \Omega(x) \cap W).$$

Indeed, suppose $\hat{y} \in \Omega(x) \setminus W$, then

$$d(y, \hat{y}) \geqslant d(y_0, \hat{y}) - d(y_0, y) > \varepsilon - \varepsilon/3 = (2/3)\varepsilon$$

and for all $z \in \Omega(x) \cap W$, $d(y, z) \leqslant d(y, y_0) + d(y_0, z) \leqslant \varepsilon/3 + \varepsilon/3 = (2/3)\varepsilon$.

Now we take any $x$ in $\tilde{U}$ with $\Omega(x) \cap \tilde{W} \neq \varnothing$. For all $z \in \Omega(x) \cap W$ and all $y \in \tilde{W}$ it follows from (3.2) that

$$d(x, \Omega^{-1}(y)) \leqslant d(x, \Omega^{-1}(y) \cap U) \leqslant K\delta^{-1}[d(y, z)].$$

Therefore,

$$d(x, \Omega^{-1}(y)) \leqslant K\delta^{-1}[d(y, \Omega(x) \cap W)] = K\delta^{-1}[d(y, \Omega(x))]$$

for all $x$ in $\tilde{U}$ with $\Omega(x) \cap \tilde{W} \neq \phi$ and for all $y$ in $\tilde{W}$. That is, $\Omega$ is $\delta$-regular around $(x_0, y_0)$. ∎

In a similar way we can also prove that $\Omega$ is $\delta$-open around $(x_0, y_0)$ if and only if $\Omega^{-1}$ is $\delta^{-1}$-*LSC* around $(y_0, x_0)$.

3.3. COROLLARY. *When* $Gr\,\Omega$ *is complete and* $\delta$ *satisfies* $\underline{\lim}_{t \downarrow 0} \delta(t)/t < \infty$ *then by Theorems* 2.2 *and* 3.2 *the following are equivalent*:

(i) $\Omega$ *is* $\delta$*-regular around* $(x_0, y_0)$;

(ii) $\Omega$ *is* $\delta$*-open around* $(x_0, y_0)$;

(iii) $\Omega$ *is* $\delta$*-approximately open around* $(x_0, y_0)$;

(iv) $\Omega^{-1}$ *is* $\delta^{-1}$*-LSC around* $(y_0, x_0)$.

One of the most important class of set-valued map is that of closed convex relations—set-valued maps with closed convex graph. Some fundamental results on continuous linear operators can be extended to the class satisfactorily. (See, for example, Borwein [6].) It is interesting to note that the following open mapping theorem for closed convex relation due to Robinson [23], Ursescu [25], and Borwein [6] can be derived directly from our theorem.

3.4. THEOREM. *Let* $X$ *and* $Y$ *be Banach spaces,* $\Omega: X \to Y$ *be a closed convex relation. If* $y_0$ *is in the core of the range of* $\Omega$*, then* $\Omega$ *is open at linear rate around* $(x_0, y_0) \in Gr\,\Omega$. *Moreover,* $\Omega^{-1}$ *is pseudo-Lipschitz around* $(y_0, x_0)$.

*Proof.* Since $y_0 \in \operatorname{core} \Omega(X)$, there exist $\delta > 0$ and $0 < \gamma < \delta$ such that

$$y_0 + \delta B_y \subset \Omega(x_0 + B_x) + \gamma B_y.$$

Now pick $\varepsilon > 0$ with $\gamma + 2\varepsilon < \delta$. For all $x$ in $B_\varepsilon(x_0)$, $y$ in $B_\varepsilon(y_0)$ with $y \in \Omega(x)$, we have, for $0 < t < 1$,

$$\begin{aligned}(1-t)\,y + t(y_0 + \delta B_y) &\subset (1-t)\,\Omega(x) + t\Omega(x_0 + B_x) + t\gamma B_y\\ &\subset \Omega[(1-t)\,x + t(x_0 + B_x)] + t\gamma B_y\\ &\subset \Omega[x + t(x_0 - x) + tB_x] + t\gamma B_y\\ &\subset \Omega[x + t(1+\varepsilon)\,B_x] + t\gamma B_y.\end{aligned}$$

Thus,

$$\begin{aligned} y + t\delta B_y &\subset \Omega[x + t(1+\varepsilon) B_x] + t(y - y_0) + t\gamma B_y \\ &\subset \Omega[x + t(1+\varepsilon) B_x] + t(\gamma + \varepsilon) B_y. \end{aligned}$$

Let $\delta' := \delta/(1+\varepsilon)$, $\gamma' := (\gamma+\varepsilon)/(1+\varepsilon)$. Then $\gamma' < \delta'$ and

$$y + t\delta' B_y \subset \Omega(x + tB_x) + t\gamma' B_y \tag{3.5}$$

for $0 < t < 1$, $y \in \Omega(x)$ and $\| y - y_0 \| \leqslant \varepsilon$, $\| x - x_0 \| \leqslant \varepsilon$. Hence $\Omega$ is approximately open at linear rate around $(x_0, y_0)$. Therefore by Theorem 2.2 $\Omega$ is open at linear rate around $(x_0, y_0)$ and by Corollary 3.3 $\Omega^{-1}$ is pseudo-Lipschitz around $(y_0, x_0)$. The proof is complete. ∎

3.5. Remark. Note that (3.5) certainly implies that for all $x$ in $B_\varepsilon(x_0)$, $y$ in $B_\varepsilon(y_0)$ with $y \in \Omega(x)$, and for all $0 < t < 1$

$$y + t\delta' B_y \subset \Omega(x + tB_y) + t\gamma' B_y.$$

It follows from Radström's cancellation lemma [22] that

$$y + (\delta' - \gamma')\, tB_y \subset \overline{\Omega(x + tB_y)}.$$

Now we use Corollary 2.3 to affirm that there exists $\varepsilon' > 0$ such that

$$\operatorname{int}[y + (\delta' - \gamma')\, tB_y] \subset \Omega(x + tB_y)$$

for all $x$ in $B_{\varepsilon'}(x_0)$, $y$ in $B_{\varepsilon'}(y_0) \cap \Omega(x)$ and $0 < t < 1$.

The above remark not only exhibits explicitly the constant of linear rate openness of $\Omega$ but also intimates the following interesting local surjectivity result for convex process due to Isac [15], Borwein [8]. Recall that a *convex process* $H$ from linear space $X$ to linear space $Y$ is a set-valued map whose graph $\mathrm{Gr}(H)$ is a convex cone in $X \times Y$.

3.6. Corollary. *Let $H$ be a closed convex process between Banach spaces $X$ and $Y$. Suppose there exist positive constants $t_0$, $M$, and $v$ such that for every $y$ in $Y$ with $\| y \| \leqslant t$, one can find $x_0$ in $X$ and $z_0 \in H(x_0)$ with the following properties*:

(i) $\| x_0 \| \leqslant M \| y \|$;

(ii) $\| y - z_0 \| \leqslant v \| y \|$.

*Then $H$ is open at linear rate with constant* $(1 - v)/M$ *around* $(0, 0)$.

*Proof.* The hypotheses give the following inclusion:

$$tB_y \subset H(MtB_x) + vtB_y \qquad (0 < t < t_0).$$

By Remark 3.5,

$$\operatorname{int}(1-v)\,B_y \subset H(MB_x),$$

or

$$(1-v)/M \operatorname{int} B_y \subset H(B_x). \quad \blacksquare$$

When $H$ is a continuous linear operator, the corollary with Urysohn's lemma applied to the restriction map from the space of bounded continuous functions on a normal space $X$ to the space of bounded continuous functions on a closed subset of $X$ yields immediately Tietze's extension theorem, as first observed by Grabiner [13].

To provide one more example illustrating approximate openness at work, we rederive Robinson's inversion theorem [24]. See also Borwein [7].

Recall that a single-valued map $G$ from Banach space $X$ to Banach space $Y$ is said to be strictly differentiable at $x_0$ if it is Fréchet differentiable at $x_0$ [5] with derivative $\nabla G(x_0)$ and

$$G(x+h) = G(x) + \nabla G(x_0)\,h + r(r, h)\|h\|,$$

where $r(x, h) \to 0$ when $x \to x_0$, $h \to 0$.

3.7. THEOREM. *Let $X$ and $Y$ be Banach spaces and let $G: X \to Y$ be strictly differentiable at $x_0$. Let $A \subset X$ be closed and convex such that $x_0 \in A$ and*

$$0 \in \operatorname{core}\{\nabla G(x_0)(A - x_0)\}, \tag{3.6}$$

*then $G|_A$ is regular at linear rate around $(x_0, G|_A(x_0))$ and $G|_A^{-1}$ is (pseudo-) Lipschitz around $(x_0, G|_A(x_0))$, where $G|_A(x) = G(x)$ if $x \in A$ and $G|_A(x) = \varnothing$ otherwise.*

*Proof.* Since $Y$ is a Banach space (3.6) is equivalent to

$$0 \in \operatorname{int}\overline{\{\nabla G|_A(x_0)(A - x_0)\}}.$$

So there exists $\delta > 0$ such that

$$\delta B_y \subset \overline{\nabla G|_A(x_0)[(A - x_0) \cap B_x]}.$$

Since $G$ is strictly differentiable at $x_0$, one can prove there exists a neighbourhood $U$ of $x_0$ with the property that for sufficient small $\gamma > 0$ and all $0 < t < t_0$, $t_0$ is fixed,

$$\overline{\nabla G|_A(x_0)[(A - x_0) \cap B_x]} \subset \frac{G|_A[x + t(A - x)] - G|_A(x)}{t} + \gamma B_y$$

for all $x \in U \cap A$. Hence,

$$\delta B_y \subset \frac{G|_A[x+t(a-x_0)] - G|_A(x)}{t} + \gamma B_y$$

for all $x$ in $U \cap A$ and $0 < t < t_0$. That is, $G|_A$ is approximately open at linear rate around $(x_0, G|_A(x_0))$ and so $G|_A$ is regular at linear rate around $(x_0, G|_A(x_0))$. Therefore $G|_A^{-1}$ is Lipschitz around $(x_0, G|_A(x_0))$. ∎

## 4. Regularity and Differentiation of Set-Valued Maps

The graph of the derivative to a smooth function is the tangent to the graph of the functon, and so Aubin introduces the derivtive of a set-valued map by defining the graph of the derivative to be the Clarke tangent cone [9] to the graph of the map.

4.1. Definition (Aubin [1]). Let $\Omega$ be a set-valued map between normed spaces $X$ and $Y$. The (Clarke) derivative of $\Omega$ at $(x_0, y_0) \in \operatorname{Gr} \Omega$ is the closed convex process $C\Omega(x_0, y_0)$ from $X$ to $Y$ whose graph is the Clarke tangent cone $C_{\operatorname{Gr} \Omega}(x_0, y_0)$ to the graph of $\Omega$ at $(x_0, y_0)$.

When $\Omega = \{G\}$ is single-valued and continuously differentiable at $x_0$, $C\Omega(x_0, \Omega(x_0)) = \nabla G(x_0)$.

Very recently, Frankowska [12] introduced the following concept of high order variations for set-valued maps.

4.2. Definition. Let $X$ and $Y$ be normed spaces and $\Omega: X \to Y$ a set-valued map. Suppose $y_0 \in \Omega(x_0)$ is given and $r \geqslant 1$ is any number. The set of $r$th order variation of $\Omega$ at $(x_0, y_0)$, is defined by

$$\Omega^r(x_0, y_0) := \liminf_{\substack{(x,y) \to (x_0, y_0) \\ y \in \Omega(x) \\ t \downarrow 0}} \frac{\Omega(x + tB_x) - y}{t^r}, \tag{4.1}$$

where lim inf is in the sense of Kuratowski [17].

High order variations for set-valued maps enjoy many interesting properties. The reader is referred to Frankowska's paper for details. For instance, the first-order variation of $\Omega$ is closely related to the image of the unit ball $B_x$ of the derivative of $\Omega$, as Frankowska observed in her paper. In fact,

$$C\Omega(x_0, y_0)(B_x) \subset \Omega^1(x_0, y_0). \tag{4.2}$$

Indeed, when $\Omega$ is single-valued and continuously differentiable at $x_0$,

$$\overline{\nabla\Omega(x_0)\, B_x} = \Omega^1(x_0, y_0). \tag{4.3}$$

However, high order variations can provide more information even when $\Omega$ is a real-valued function.

4.3. Examples. (i) Let $f(x) := x^{2n+1}$ for $x$ in $\mathbb{R}$ and a positive integer $n$. Then $B_x = [-1, 1]$ and

$$\begin{aligned}
f^{2n+1}(0,0)(B_x) &= \liminf_{\substack{x \to 0 \\ t \downarrow 0}} \frac{(x + tB_x)^{2n+1} - x^{2n+1}}{t^{2n+1}} \\
&= \liminf_{\substack{x \to 0 \\ t \downarrow 0}} \{(z + B_x)^{2n+1} - z^{2n+1}\} \qquad (z = x/t) \\
&= \bigcap_{z \in \mathbb{R}} \{(z + B_x)^{2n+1} - z^{2n+1}\} \\
&= \bigcap_{z \in \mathbb{R}} [(z-1)^{2n+1} - z^{2n+1}, (z+1)^{2n+1} - z^{2n+1}] \\
&= \left[-\left(\frac{1}{4}\right)^n, \left(\frac{1}{4}\right)^n\right].
\end{aligned}$$

In fact,

$$f^r(0,0)(B_x) = \begin{cases} 0 & \text{if } \alpha < 2n+1 \\ [-(\frac{1}{4})^n, (\frac{1}{4})^n] & \text{if } \alpha = 2n+1 \\ \mathbb{R} & \text{if } \alpha > 2n+1. \end{cases}$$

(ii) Let $g(x) := x^{2n}$, for $x$ in $\mathbb{R}$ and a positive integer $n$. Then,

$$\begin{aligned}
g^{2n}(0,0)(B_x) &= \bigcap_{z \in \mathbb{R}} \{(z + B_x)^{2n} - z^{2n}\} \\
&= \bigcap_{z \in \mathbb{R}} [0, \max(z \pm 1)^n - z^n] \\
&= [0, 1].
\end{aligned}$$

Also, we have

$$g^r(0,0)(B_x) = \begin{cases} 0 & \text{if } r < 2n \\ [0, 1] & \text{if } r = 2n \\ \mathbb{R} & \text{if } r > 2n. \end{cases}$$

When $X$ is a Banach space, $Y$ is a finite dimensional space and $\Omega$ is a closed set-valued map with $y_0 \in \Omega(x_0)$ being given, Frankowska [12] proves that if for some integer $r \geqslant 1$

$$0 \in \operatorname{int} \Omega^r(x_0, y_0), \tag{4.4}$$

then $\Omega$ is open around $(x_0, y_0)$.

It turns out that in this setting (4.4) is equivalent to our definition of regularity of $\Omega$ at rate $r$ around $(x_0, y_0)$ and its equivalences.

4.4. THEOREM. *Let $X$ be a complete metric space and $Y$ a finite dimensional space and let $\Omega$ be a closed set-valued map and $(x_0, y_0) \in \operatorname{Gr} \Omega$ be given. Then the following are equivalent for $r \geqslant 1$.*

(i) $0 \in \operatorname{int} \Omega^r(x_0, y_0)$;

(ii) *$\Omega$ is regular at rate $r$ around $(x_0, y_0)$;*

(iii) *$\Omega$ is open at rate $r$ around $(x_0, y_0)$;*

(iv) *$\Omega$ is approximately open at rate $r$ around $(x_0, y_0)$;*

(v) *$\Omega^{-1}$ is pseudo-Hölder at rate $r$ around $(y_0, x_0)$.*

*Proof.* We first demonstrate that (i) implies (iv).

Let $B_y$ be the unit ball in $y$. Fix $\varepsilon$ in $(0, 1/2)$. Since $B_y$ is (pre)compact, there exists a finite set $F := \{p_1, ..., p_k\} \subset Y$ such that

$$B_y \subset F + \varepsilon B_y. \tag{4.5}$$

As $0 \in \operatorname{int} \Omega^r(x_0, y_0)$, we can choose some $\delta > 0$ such that for every $p$ in $F$, $\delta p \in \Omega^r(x_0, y_0)$. From the definition of the $r$th variation, we see that for $v := 2\varepsilon\delta < \delta$ there exist a neighbourhood $U$ of $x_0$ and a neighbourhood $W$ of $y_0$ such that for all $t > 0$ with $B_t(x_0) \subset U$,

$$\delta p \in \frac{\Omega[B_t(x)] - y}{t^r} + \frac{\gamma}{2} B_y$$

if $x \in U$ and $y \in \Omega(x) \cap W$. Hence

$$\delta F \subset \frac{\Omega[B_t(x)] - y}{t^r} + \frac{\gamma}{2} B_y.$$

Now

$$\begin{aligned} \delta B_y &\subset \delta F + \delta\varepsilon B_y \\ &\subset \frac{\Omega[B_t(x)] - y}{t^r} + \frac{\gamma}{2} B_y + \frac{\gamma}{2} B_y \\ &\subset \frac{\Omega[B_t(x)] - y}{t^r} + \gamma B_y \end{aligned}$$

for all $x$ in $U$, $y$ in $\Omega(x) \cap W$ and $t>0$ with $B_t(x_0) \subset U$. Therefore $\Omega$ is approximately open at rate $r$ around $(x_0, y_0)$.

On the other hand, if $\Omega$ is open at rate $r$ around $(x_0, y_0)$ then it follows directly that (i) holds. Thus (iv) is equivalent to (i), (ii), and (iii). ∎

The following example illustrates that a map can be open around some point while not being open at any rate $r>0$ around the point. Thus, Theorem 4.4 strengthens Frankowska's open mapping principle [12].

4.5. Example. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x)=\begin{cases} \exp\left(-\dfrac{sgn(x)}{|x|}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x=0. \end{cases}$$

Then

$$f^{-1}(y)=\begin{cases} \dfrac{-y}{|y| \ln |y|} & \text{if } y \neq 0 \\ 0 & \text{if } y=0 \end{cases}$$

is continuous around (0, 0). Hence $f$ is open around (0, 0). However,

$$\lim_{x \to 0} \left|\frac{f(x)}{x^r}\right| = \lim_{x \to 0} \frac{e^{-1/|x|}}{|x|^r} = 0$$

for all $r>0$. Thus $f^r(0, 0)=\{0\}$ for all $r>0$ which violates (4.4). So $f$ is not open at any rate $r>0$ around (0, 0); $f^{-1}$ is continuous, but not Hölder at any rate around (0, 0).

Our next example shows that a map being open at some point is weaker than it being open around the point.

4.6. Example. (i) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x)=\begin{cases} \dfrac{\pm 1}{2^{2n+1}} & \text{if } x \in \pm\left[\dfrac{1}{2^{2n+1}}, \dfrac{1}{2^{2n}}\right] \\ \pm\left(\dfrac{3}{2}x-\dfrac{1}{2^{2n+2}}\right) & \text{if } x \in \pm\left(\dfrac{1}{2^{2n+2}}, \dfrac{1}{2^{2n+1}}\right) \end{cases}$$

for $n=0, 1, 2, \ldots$ .

Then $f$ is open at 0 but not open around (0, 0).

Penot [20′] shows that a set-valued map $\Omega$ is open around at linear rate $(x_0, y_0)$ with $y_0 \in \Omega(x_0)$ if and only if $\Omega^{-1}$ is pseudo-Lipschitz around $(y_0, x_0)$ and if and only if $\Omega$ is regular at linear rate around $(x_0, y_0)$. Our next example shows that in general our approximate openness is not the same as openness.

(ii) Let $\Omega: \mathbb{R} \to \mathbb{R}^2$ be defined by $\Omega(x) = C$ for all $x \in \mathbb{R}$, where $C$ is the union of the unit balls centred at $(-1, 0)$, $(1, 0)$, respectively. Then $0 \in \Omega(0)$ and

$$\lim_{t \downarrow 0} \frac{\Omega(tB) - 0}{t} = t^{-1}(C - 0) = \mathbb{R}^2.$$

One can prove that $\Omega$ is approximately open at linear rate at $(0, 0)$. However, as the origin $(0, 0)$ is in the boundary of $C$, $\Omega$ is not open at $(0, 0)$.

Finally, we rederive from our previous results an inverse mapping theorem for set-valued maps proved recently by Aubin and Frankowska [3].

4.7. THEOREM (Aubin and Frankowska [3]). *Suppose $\Omega$ is a closed set-valued map from a Banach space $X$ into a finite dimensional space $Y$. Let $(x_0, y_0)$ in the graph of $\Omega$ be given. If the (Clarke) derivative of $\Omega$ at $(x_0, y_0)$ is surjective, then $\Omega^{-1}$ is pseudo-Lipschite around $(y_0, x_0)$.*

*Proof.* The surjectivity of the closed convex process $C\Omega(x_0, y_0)$ is equivalent to $0 \in \text{core}[C\Omega(x_0, y_0)(X)]$. By Theorem 3.4 there exists $\delta > 0$ such that

$$\delta B_y \subset C\Omega(x_0, y_0)\, B_x.$$

Hence, Theorem 4.4 shows that $\Omega^{-1}$ is pseudo-Lipschitz around $(y_0, x_0)$. ∎

The same technique can be used to prove inverse mapping theorems in Frechet spaces. The reader is referred to the forthcoming thesis [26].

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